Model

- The model contains:
  - A set $\mathcal{N}$ of $N \in \mathbb{N}$ agents divided into $k \in \mathbb{N}$ subsets $T_1, T_2, ..., T_k$ called types
  - A set $\mathcal{M}$ of $M \in \mathbb{N}$ items divided into $l \in \mathbb{N}$ subsets $B_1, B_2, ..., B_l$ called blocks
  - A matrix $\lambda \in [1,k] \times [1,l]$ of quotas

- An allocation $A$ is a function that maps every agent $i$ to a set $A(i)$ of items. An allocation is valid if and only if:
  - Each agent receives at most one item.
  - Agents do not share item.
  - All items are assigned.
  - Quotas are respected: for each type $p$ and each block $q$, the number of items from the block $q$ assigned to agents of type $p$ is at most $\lambda_{p,q}$.

Utility

- Utility function
In an allocation $A$, the utility $u_i(A)$ of agent $i$ is:

$$u_i(A) = u_i^l(A) + \varphi \times u_i^N(A)$$

Where:

- $u_i^l(A) \in [0,1]$ is the utility that the agent $i$ has for the item received in the allocation $A$.
- $u_i^N(A) \in [0,1]$ is the utility that the agent $i$ receives from its neighbours, and is equal to the proportion of agents of the same type as $i$ who have been allocated item of the same block as $i$. More formally:

$$u_i^N(A) = \frac{\sum_{j \in \mathcal{N} \setminus \{i\} : A(j) \in B(A(i))} |T(i), T(j)|}{|B(A(i))|}$$

Where $T(i)$ is the type of $i$ and $B(A(i))$ is the block of the item allocated to $i$.

- If no item is assigned to an agent, the utility for this agent is 0.

- **Particular cases**
  - An agent is item-based if $\varphi = 0$ for its utility.
  - An agent is neighbourhood-based if it receives an utility of 0 for all items.
  - The social welfare, or global utility, is the sum of the utilities of all the agents.

Stability

- **Swap-Deal**
- A swap-deal between two agents $i$ and $j$ is said to be improving if and only if: $u_i(A(i)) < u_i(A(j))$ and $u_j(A(j)) < u_j(A(i))$

- **Stability**: An allocation is stable if there is no improving swap.

- **Price of Stability**

Let $U_{OPT}$ be the social welfare of the allocation maximizing the global utility, and let $U_{STABLE}$ be the social welfare of the best stable allocation.

We define the Price of Stability (PoS) as:

$$PoS = \frac{U_{OPT}}{U_{STABLE}}$$

- Proposition: PoS = 1 when all the agents are item-based or when all the agents are neighbourhood-based. PoS $\geq 1$ in the general case.

Sequential Mechanism

- **How does it work?**
  - In some random order, the agents sequentially pick the items that maximize their utilities at the time of their selection, while respecting the diversity constraints.
  - If an agent cannot be assigned to any item because of the quotas, it is skipped.
  - The algorithm ends when all agents are either assigned or unable to be assigned to any remaining item.

- **Propositions**
  - The Sequential mechanism does not always return a valid allocation.
  - The Sequential mechanism does not always return a stable allocation, unless all agents are item-focused.

- The swap mechanism will always reach a stable outcome.
- The worst-case error for the swap algorithm is unbounded in the general case.

Sequential and Swap Mechanism for Public Housing Allocation with Quotas and Neighbourhood-Based Utilities

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