Inverse problem: Gaussian Processes and Backus Gilbert

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Inverse problems

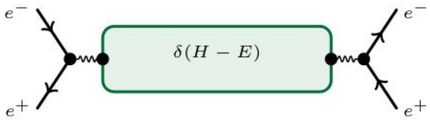
- Computing the spectral densities associated to a lattice correlators

$$C(t) = \int_0^\infty dE \, \rho(E) e^{-tE}$$

- Allow access to: inclusive decay rates, scattering amplitudes, finite-volume spectrum . . .

- Ill-posed in presence of error on the data.

- This talk: two approaches, Backus Gilbert (BG) and Gaussian Processes (GP).



Generalities

Regardless of the approach

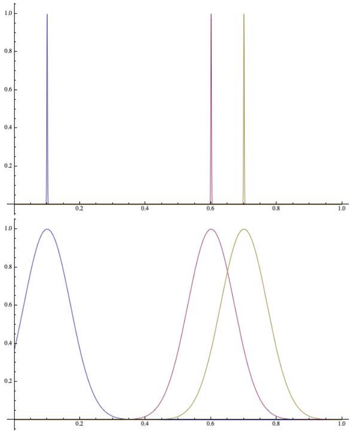
- The spectral density must be smeared, so that it is a smooth function even in a finite volume

$$\rho_\sigma(\omega) = \int_0^\infty dE\, \Delta_\sigma(\omega, E)\rho(E)$$

- Must use a regularisation. Ideally results should not depend on anything unphysical (within statistical errors).

- Reliable estimates of the systematics are necessary.

- More in Max Hansen's talk tomorrow



Backus Gilbert

- (Hansen Lupo Tantalo 2019) Solve directly for a smeared spectral density $\rho_\sigma(E)$. Aim to find g_t that span the targer kernel.

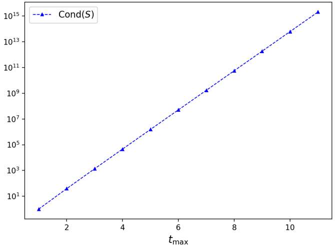
$$\sum_{t=1}^{\infty} g_t\, e^{-tE} = \Delta_\sigma(\omega, E) \implies \rho_\sigma(E_\star) = \sum_{t=1}^{\infty} g_t C(t)$$

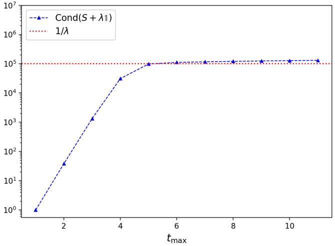
- g_t obtained minimising

$$\underbrace{\int_0^\infty dE \left| \sum_{t=1}^{t_{\max}} g_t e^{-tE} - \Delta_\sigma(\omega, E) \right|^2}_{\text{Provides exact solution}} + \lambda \underbrace{\vec{g} \cdot \mathrm{Cov}_d \cdot \vec{g}}_{\text{Regularises}}$$

- Amounts to invert the matrix T. The first terms express the ill defined nature of the problem.

$$T^{BG} = \int_0^\infty dE\; e^{-(t+r)E} + \lambda\, \mathrm{Cov}_d$$





Bayesian Inference with Gaussian Processes

- Aim for a probability distribution over a functional space of possible spectral densities

- GP: consider the stochastic field $\mathcal{R}(E)$ Gaussian-distributed around the prior value $\rho_{\text{mod}}(E)$ with covariance $\mathcal{S}(E, E')$.

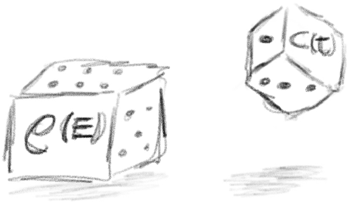
$$\mathcal{GP}\left(\mathcal{R}(E) - \rho_{(\text{mod})}(E), \mathcal{S}(E, E')\right)$$

- Similarly, assume that observational noise is Gaussian: $\eta(t)$

$$\mathbb{G}\left(\eta, \text{Cov}_d\right) = \exp\left(-\frac{1}{2}\vec{\eta}^T \; \text{Cov}_d \; \vec{\eta}\right)$$

- The relation between the two induces a correlation.

$$C_{\text{obs}}(t) = \int dE \, e^{-tE}\mathcal{R}(E) + \eta(t)$$



Gaussian Processes vs Backus Gilbert

- Given the previous relation, the Bayes theorem yields the joint distribution for $\rho(\omega)$ and $C_{\rm obs}$ (Pawlowski et al. 2020, Del Debbio et al. 2021)

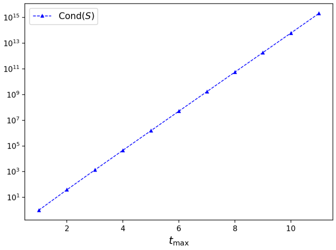
- Central value of posterior distribution gives a prediction for $\rho(\omega)$

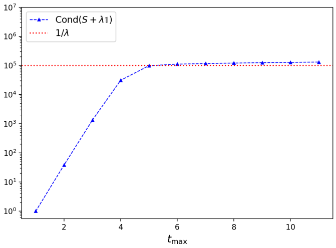
$$\rho(\omega) = \rho_{\rm mod}(\omega) + \int dE\ e^{-tE} \mathcal{S}(E,\omega)\ (T^{GP})^{-1}_{tr}\ \ (C_{\rm obs}(r) - C_{\rm mod}(r))$$

$$T^{GP} = \int dE_1\ dE_2\ e^{-tE_1}\ \mathcal{S}(E_1, E_2)\ e^{-rE_2} + {\rm Cov}_d$$

(1) Compare the matrix inverted with Backus Gilbert. Numerically both approaches have the same regularisation.

$$T^{BG} = \int_0^\infty dE\ e^{-(t+r)E} + \lambda\ {\rm Cov}_d$$



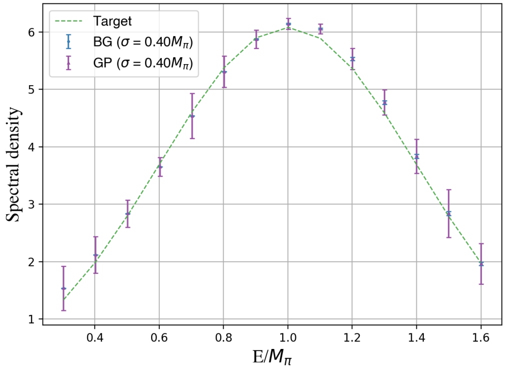


Correspondence

- BG (Hansen Lupo Tantalo 2019): look for a solution smeared with a Gaussian.
- Look for the probability distribution of a spectral density smeared with a Gaussian. Requires to generalise the previous equations. (Valentine, Sambridge 2020)
- Set up GP with a specific prior: the probability distribution for the smeared spectral density is centered around ρ_σ^{BG}:

$$S(E_1, E_2) = \frac{\delta(E_1 - E_2)}{\lambda} \quad , \quad \rho_{\text{mod}} = 0$$

- Only difference: estimate of the statistical error.



Conclusions and outlook

$$S(E_1, E_2) = \frac{\delta(E_1 - E_2)}{\lambda}, \quad \rho_{\rm mod} = 0$$

- BG and GP both provide an answer in terms of a smeared version of the unknown function.
- By looking for an answer smeared with a given function we obtain the same prediction
- BG side: ideally changes in algorithmical parameters (e.g. λ) cannot affect physics $\implies$ results must fluctuate within statistical error.
- In the Bayesian language: ideally the result should not depend on the priors within statistical errors.
- In progress: relation between estimates for statistical error, study of systematics, . . .