Non-Perturbative Renormalisation with Interpolating Momentum schemes

Nicolas Garron (Liverpool Hope University)

In collaboration with

Caroline Cahill, Martin Gorbahn, John Gracey, Paul Rakow

(University of Liverpool)

UKLFT Annual Meeting

26/27 May 2022

In collaboration with

- Caroline Cahill, Martin Gorbahn, John Gracey, Paul Rakow (Liverpool)
- Thanks to RBC-UKQCD collaborations, in particular Tobias Tsang (Southern Danemark)
- Thanks to Holger Perlt (Leipzig) and QCDSF
- Thanks to CalLat (Henry Jose Monge Camacho, Amy Nicholson, André Walker-Loud)

Proceedings and preprint (submitted to PRD)

- https://arxiv.org/abs/2202.04394
- https://arxiv.org/abs/2112.11140

The RBC & UKQCD collaborations

UC Berkeley/LBNL
Aaron Meyer

BNL and BNL/RBRC
Yasumichi Aoki (KEK)
Peter Boyle (Edinburgh)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Christopher Kelly
Meifeng Lin
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

CERN
Andreas Jüttner (Southampton)

Columbia University
Norman Christ
Duo Guo
Yikai Huo
Yong-Chull Jang
Joseph Karpie
Bob Mawhinney
Ahmed Sheta
Bigeng Wang
Tianle Wang
Yidi Zhao

University of Connecticut
Tom Blum
Luchang Jin (RBRC)
Michael Riberdy
Masaaki Tomii

Edinburgh University
Matteo Di Carlo
Luigi Del Debbio
Felix Erben
Vera Gülpers
Tim Harris
Raoul Hodgson
Nelson Lachini
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
James Richings
Azusa Yamaguchi
Andrew Z.N. Yong

KEK
Julien Frison

Liverpool Hope / UoL
Nicolas Garron

Michigan State University
Dan Hoying

Milano Bicocca
Mattia Bruno

Peking University
Xu Feng

University of Regensburg
Davide Giusti
Christoph Lehner (BNL)

University of Siegen
Matthew Black
Oliver Witzel

University of Southampton
Nils Asmussen
Alessandro Barone
Jonathan Flynn
Ryan Hill
Rajnandini Mukherjee
Chris Sachrajda

University of Southern Denmark
Tobias Tsang

Stony Brook University
Jun-Sik Yoo
Sergey Syritsyn (RBRC)

On the importance of the renormalisation scheme

based on RBC-UKQCD 2010-now

. . . [NG Hudspith Lytle'16] , [Boyle NG Hudspith Lehner Lytle '17] [. . . Kettle, Khamseh, Tsang 17-19]

Four-fermion operators

Let us consider the four-quark operators that occur for example in BSM neutral $\langle \bar{K}^0 | O^{\Delta F=2} | K^0 \rangle$

Similar matrix elements in the heavy physics contribution to neutrinoless double beta decay (in the $\pi^- \to \pi^+$ transitions)

[Nicholson, Berkowitz, Monge-Camacho, Brantley, NG, Chang, Rinaldi, Clark, Joo, Kurth, Tiburzi, Vranas, Walker-Loud '18]

Four-fermion operators

$$
(27,1) \qquad O_1^{\Delta S=2} = \gamma_\mu \times \gamma_\mu + \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5
$$

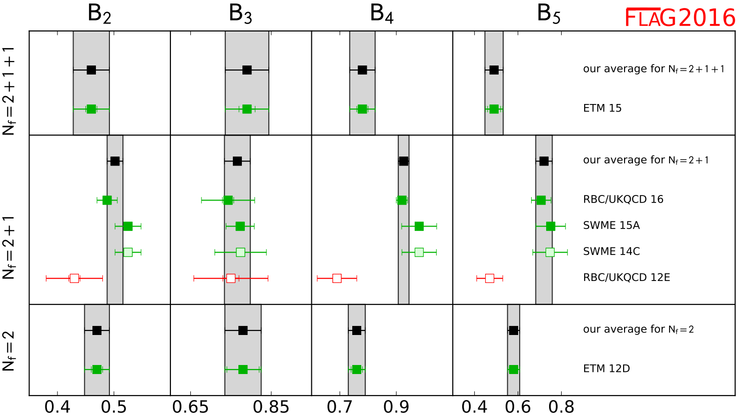
$$
(8,8) \qquad \begin{cases} O_2^{\Delta s=2} = \gamma_\mu \times \gamma_\mu - \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5 \\ O_3^{\Delta s=2} = 1 \times 1 - \gamma_5 \times \gamma_5 \end{cases}
$$

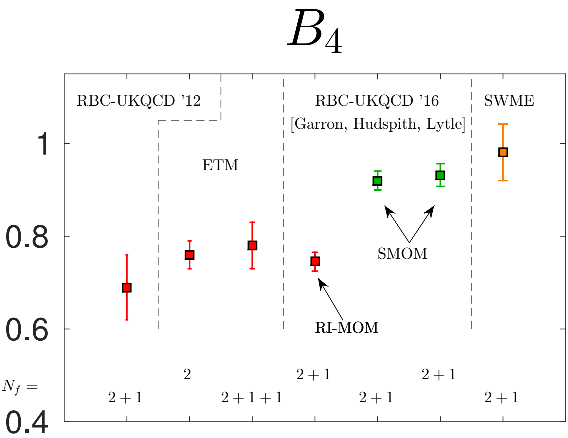
$$
(6,\bar{6}) \qquad \begin{cases} O_4^{\Delta s=2} = 1 \times 1 + \gamma_5 \times \gamma_5 \\ O_5^{\Delta s=2} = \sigma_{\mu\nu} \times \sigma_{\mu\nu} \end{cases}
$$

So the renormalisation matrix has the form

$$
\mathcal{Z}^{\Delta S=2} = \begin{pmatrix} \mathcal{Z}_{11} & & & & \\ & \mathcal{Z}_{22} & \mathcal{Z}_{23} & & \\ & \mathcal{Z}_{32} & \mathcal{Z}_{33} & & \\ & & & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ & & & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}
$$

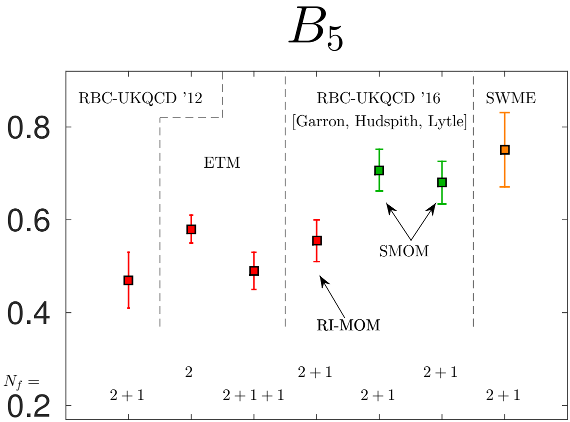
BSM kaon mixing - Results



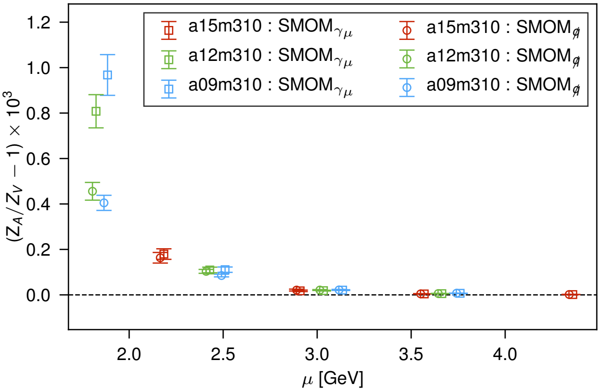


B_4
RBC-UKQCD '12
ETM
RBC-UKQCD '16
[Garron, Hudspith, Lytle]
SWME
SMOM
RI-MOM
1
0.8
0.6
0.4
$N_f =$
2 + 1
2
2 + 1 + 1
2 + 1
2 + 1
2 + 1
2 + 1

BSM kaon mixing - Results



Another example Z_A/Z_V



[C Chang, A Nicholson, E Rinaldi, E Berkowitz, N G, D Brantley, H Monge-Camacho, C Monahan, C Bouchard, M Clark, B Joó, T Kurth, K Orginos, P Vranas, A Walker-Loud, Nature 558 (2018)]

Non Perturbative Renormalisation (NPR)

Reminders - general strategy

- We use Lattice QCD to compute the non-perturbative effects of the strong interaction in well-defined manner.

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- As usual for a renormalisable QFT, in order to make some predicition we have to regularise and renormalise the theory.
- Inverse lattice spacing a plays the role of the UV regulator.
- Have to remove the UV divergences before taking the continuum limit $a \longrightarrow 0$.

Reminders - general strategy

1. First step: remove the divergences

   For a generic composite operator $Q^{bare}(a)$ which renormalises multiplicatively, determine the Z-factor such that

   $$Q^{scheme}(\mu, a) = Z^{scheme}(\mu, a) Q^{bare}(a)$$

   has well-defined continuum limit.

   This step can be done non-perturbatively.

2. Second step: match to phenomenology (e.g. $\overline{\mathrm{MS}}$), This step has to be done in (continuum) perturbation theory .

$$Q_i^{scheme}(\mu, 0) \longrightarrow Q^{\overline{\mathrm{MS}}}(\mu) = (1 + r_1 \alpha_S(\mu) + r_2 \alpha_S^2(\mu) + \ldots) Q^{scheme}(\mu, 0)$$

Non-Perturbative Renormalisation (NPR)

There are two popular methods for the non-perturbative determination of the Z-factors

- Schrödinger Functional (SF)
- Rome-Southampton: RI/MOM, RI/MOM', RI/SMOM, RI/mSMOM, ...

Here I am talking about extensions of the latter.

Original setup [Martinelli et al '95].

Consider for example a quark bilinear build from quark propagators.

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- Off-shell: Send $m_q \longrightarrow 0$ (chiral limit) and the renormalisation scale is given by some momentum $\mu = \sqrt{p^2}$.
- Gauge fixed (Landau).

Prescription: one requires some amputated Green function(s) to be finite.

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Several extensions and improvement, most notably

- Non-exceptional kinematics (SMOM) [RBC, RBC-UKQCD, Sturm et al., Lehner and Sturm, Almeida and Sturm, Gorbahn and Jäger, Gracey, ...]
- Momentum sources (QCDSF)
- Twisted boundary conditions [many references !]
- Massive momentum scheme [Boyle, Del Debbio and Khamseh, 2016]
- Step scaling [Alpha, RBC-UKQCD, ...]

Example: quark bilinear

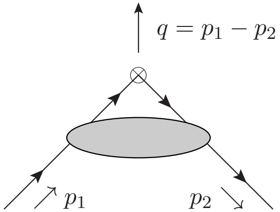
Consider a quark bilinear $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$, where $\Gamma = \mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle G_2(x_2, 0) \Gamma G_1(0, x_1) \rangle$$

In Fourier space $G(p) = \sum_x G(x, 0) e^{ip.x}$ and $G(-p) = \gamma_5 G(p)^\dagger \gamma_5$

$$V(p_2, p_1) = \langle G_2(-p_2) \Gamma G_1(p_1)^\dagger) \rangle$$



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Amputated Green function

$$\Pi(p_2, p_1) = \langle G_2(p_2)^{-1} \rangle \langle G_2(p_2) \Gamma G_1(p_1)^\dagger) \rangle \langle (G_2(p_1)^{\dagger^{-1}}) \rangle$$

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Rome Southampton original scheme (RI/MOM), $p_1 = p_2 = p$ and $\mu = \sqrt{p^2}$

$$\frac{Z_\Gamma}{Z_q}(\mu, a) \times \lim_{m \to 0} \mathrm{Tr}(\Gamma\, \Pi(p, p))_{\mu^2 = p^2} = \mathrm{Tree}$$

Rome-Southampton windows

Ideally, in order to keep the discretisation effects under control [G. Martinelli et al 94]

$$\mu \ll 1/a$$

and to apply perturbation theory

$$\Lambda_{\mathrm{Q}CD} \ll \mu$$

In practice, might be tight if $1/a \sim 2\,\mathrm{G}eV$

Rome-Southampton windows

Imagine you have computed the hadronic matrix elements on a coarse lattice

$$1/a_{coarse} \sim 1.4\,\mathrm{GeV}$$

and want the renormalised answer at a scale of $\mu \sim 3\,\mathrm{GeV}$

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Improvement inspired by step-scaling methods à la [Alpha collaboration]

$$Z(a_{coarse},\mu) = \lim_{a_{fine}\to 0} \left\{ Z(a_{fine},\mu) Z^{-1}(a_{fine},\mu_0) \right\} \times Z(a_{coarse},\mu_0)$$

Where

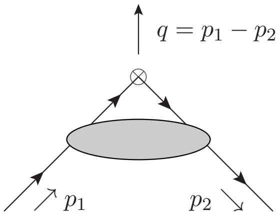
- μ_0 is a lower scale, eg $\mu_0 \sim 1\,\mathrm{Ge}V$
- the running is computed on finer lattices and extrapolated to the continuum

[Arthur and Boyle '10], [Arthur, Boyle, N.G., Kellu, Lytle '11]

Crucial for example in the computation of $K \to \pi\pi$ decays [RBC-UKQCD '11-']

RI/MOM vs RI/SMOM

Kinematics



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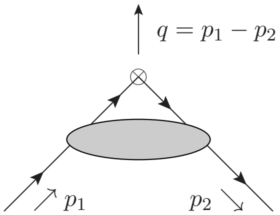

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  Lead to IR poles, for example in $1/\mu^2$
- In RI/SMOM we have

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2 = (p_1 - p_2)^2$$

Improved IR behaviour [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, ...]

Pole subtraction

- The Green functions might suffer from IR poles, $\sim 1/p^2$, or $\sim 1/m_\pi^2$ which can pollute the signal
- In principle these poles are suppressed at high μ but they appear to be quite important at $\mu \sim 3$ GeV for some quantities which allow for pion exchanges
- The traditional way is to "subtract " these contamination by hand

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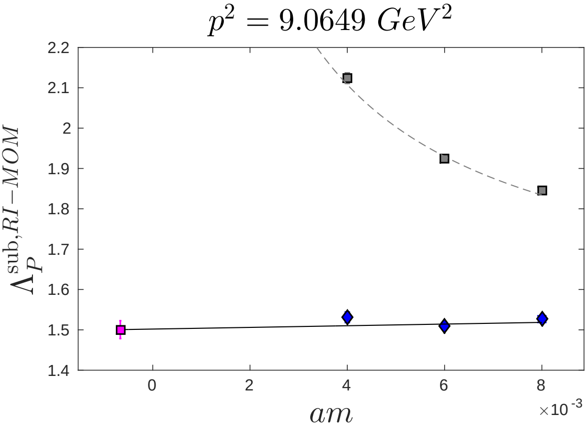
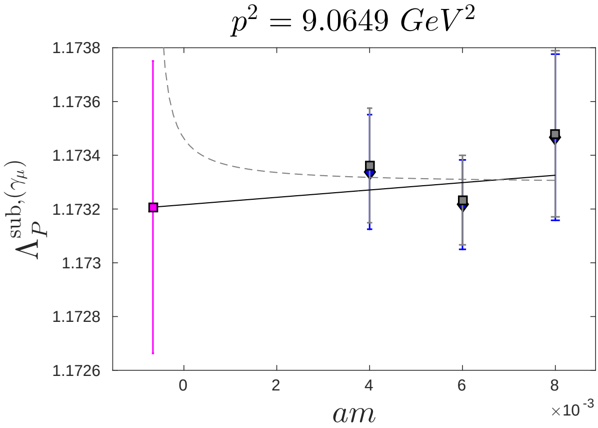
- In principle these poles are suppressed at high μ but they appear to be quite important at $\mu \sim 3$ GeV for some quantities which allow for pion exchanges
- The traditional way is to "subtract " these contamination by hand
- However these contaminations are highly suppressed in a SMOM scheme, with non-exceptional kinematics
- We argue that this pion pole subtractions is not-well under control and that schemes with exceptional kinematics should be discarded

Pole subtraction (I)




$p^2 = 9.0649\ GeV^2$
1.1738
1.1736
1.1734
1.1732
1.173
1.1728
1.1726
$\Lambda_P^{\mathrm{sub},(\gamma_\mu)}$
0
2
4
6
8
$\times 10^{-3}$
am

In [Boyle, NG, Hudspith, Lehner, Lytle, '17 (1708.03552)] we did a careful study and argued that the disagreement observed between different computations is due to the renormalisation procedure. We argued that the pole-subtraction procedure is prone to systematic errors.

SMOM and IMOM

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

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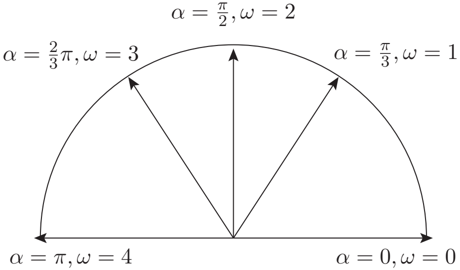
- We are now studying a generalisation (see also [Bell and Gracey, Perlt])

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2, \quad (p_1 - p_2)^2 = \omega\mu^2 \text{ where } \omega \in [0, 4]$$

Note that $\omega = 0 \leftrightarrow RI/MOM$ and $\omega = 1 \leftrightarrow RI/SMOM$

IMOM schemes

α is the angle between p_1 and p_2



$$\omega = 2(1 - \cos\alpha)$$

Implementation (1)

We want to achieve $p_1^2 = p_2^2 \equiv \mu^2\,, \quad q^2 = (p_1 - p_2)^2 = \omega\mu^2\ ,$

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One possibility, for example [QCDSF'17]

$$p_1 = \frac{2\pi}{L}\,(m,m,m,m)\ , \quad p_2 = \frac{2\pi}{L}\,(-m,-m,-m,m)$$

$$\Rightarrow q = \frac{2\pi}{L}(2m,2m,2m,0)$$

gives

$$\mu^2 = \left(\frac{2\pi}{L}\right)^2 4m^2\ , \text{and} \quad q^2 = 3\mu^2$$

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The number of $-$ signs in p_2 gives the value of $\omega = 0, 1, \ldots, 4$.

Implementation (2)

Another possibility is to take advantage of twisted boundary conditions, say take

$$p_1 = \frac{2\pi}{L}\,(l,0,0,0) \qquad p_2 = \frac{2\pi}{L}\,(m,n,0,0)$$

$$\Rightarrow q = \frac{2\pi}{L}(l-m,-n,0,0)$$

And for each pair of desired (μ,ω), just need to solve

$$\begin{aligned} \mu &= 2\pi/L \\ l^2 &= m^2+n^2 \\ \omega l^2 &= (l-m)^2+n^2 \end{aligned}$$

Definitions

We call Λ_Γ the projected-amputated Green function, normalised by its tree value

For example $$\Lambda_S = \frac{1}{12}\mathrm{Tr}\left(\Pi_S\right) \ .$$

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We define $Z_m = 1/Z_S$ and compute the Z-factors for the scalar density

$$\left(\frac{Z_S}{Z_q}(\mu,\omega)\right)^{\mathrm{IMOM}} \times \left(\Lambda_S\right)_{q^2=\omega\mu^2} = 1$$

For Z_q we use the vector current

$$\left(\frac{Z_V}{Z_q}(\mu,\omega)\right)^{\mathrm{IMOM}-\gamma_\mu} \times \left(\Lambda_V^{(\gamma_\mu)}\right)_{q^2=\omega\mu^2} = 1$$

and

$$\left(\frac{Z_V}{Z_q}(\mu,\omega)\right)^{\mathrm{IMOM}-\not{q}} \times \left(\Lambda_V^{(\not{q})}\right)_{q^2=\omega\mu^2} = 1$$

Projectors

The difference between $\mathrm{IMOM}-\gamma_\mu$ and $\mathrm{IMOM}-\not{q}$ lies in the projector

$$\begin{aligned}\Lambda_V^{(\gamma_\mu)} &= \frac{1}{48}\mathrm{Tr}\left(\gamma_\mu \Pi_{V^\mu}\right)\\ \Lambda_V^{(\not{q})} &= \frac{q^\mu}{12q^2}\mathrm{Tr}\left(\not{q}\,\Pi_{V^\mu}\right)\end{aligned}$$

Ward-Takahashi identities (I)

In the continuum we have

$$\begin{aligned} q_\mu \Pi_{V^\mu}(p_1, p_2) &= -i(G^{-1}(p_2) - G^{-1}(p_1))\,, \\ q_\mu \Pi_{A^\mu}(p_1, p_2) &= 2im\Pi_P(p_1, p_1) - i\left(\gamma_5 G^{-1}(p_2) + G^{-1}(p_1)\gamma_5\right)\,, \end{aligned}$$

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and using the decomposition

$$G^{-1}(p) = i\not{p}(1 + \Sigma^V) + m(1 + \Sigma^S) \,,$$

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and using the decomposition

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leads to

$$\Lambda_V^{(\not{q})} = \frac{i}{12}\mathrm{Tr}\left(1+\Sigma^V\right)\;,$$

and therefore expect $Z_V^{(\not{q})}$ to be ω-independent.

Ward-Takahashi identities (II)

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If chiral symmetry is explicitly broken, one can impose the VWI and use it as a renormalisation condition.

Here we employ Domain-Wall fermions with good chiral-flavour symmetry. We can therefore use the VWI as a consistency check of our strategy.

Numerical results

Simulation

We use RBC-UKQCD ensembes, IW, 2+1 Domain-Wall fermions

We have two lattice spacings:

$$a^{-1} = 1.785(5)\ \text{GeV} \quad (24^3) \tag{1}$$
$$a^{-1} = 2.383(9)\ \text{GeV} \quad (32^3), \tag{2}$$

sea quark masses, $am = 0.005,\ 0.010,\ 0.020$ for the $24^3 \times 64 \times 16$ lattice

and $am = 0.004,\ 0.006,\ 0.008$ for the $32^3 \times 64 \times 16$ lattice.

We take the chiral limit on each lattice spacing using the values

$$am_{res} = 0.003152(43) \quad (24^3), \tag{3}$$
$$am_{res} = 0.0006664(76) \quad (32^3). \tag{4}$$

Our values for Z_V are

$$Z_V = Z_A = 0.71651(46) \quad (24^3), \tag{5}$$
$$Z_V = Z_A = 0.74475(12) \quad (32^2). \tag{6}$$

Results

Non-perturbative scale evolution (running), taking the continuum limit

$$\sigma(\mu,\omega,\mu_0,\omega_0) = \lim_{a^2\to 0} \frac{Z(\mu,\omega)}{Z(\mu_0,\omega_0)}$$

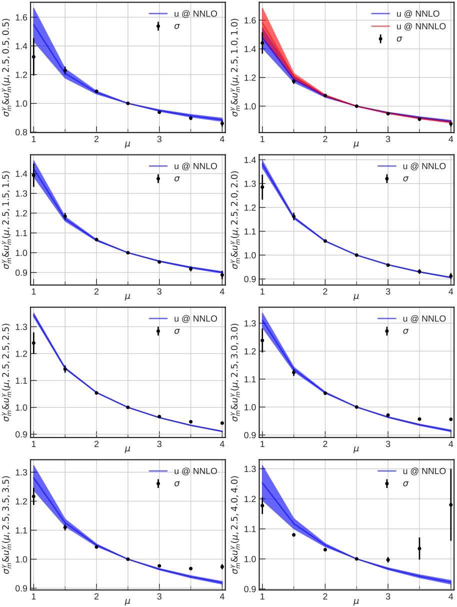
We have computed the perturbative prediction at NNLO, $U(\mu,\omega,\mu_0,\omega_0)$

In the next slides, I show some plots for the ratios

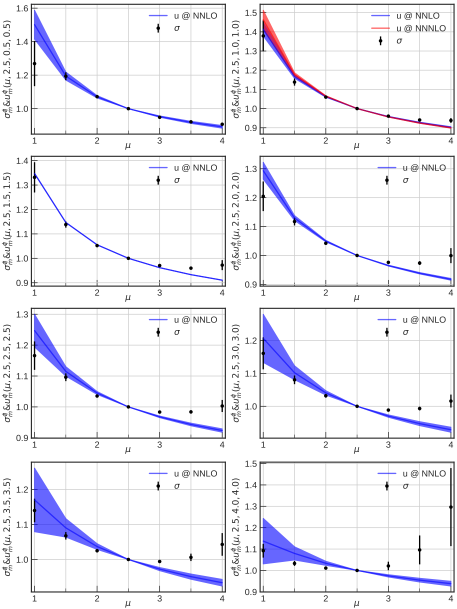
$$\frac{\sigma(\mu,\omega,\mu_0,\omega_0)}{U(\mu,\omega,\mu_0,\omega_0)}$$

for fixed μ_0,ω_0 and various order in PT

Results for $Z_m^{(\gamma_\mu)}$



Results for $Z_m^{(\not q)}$



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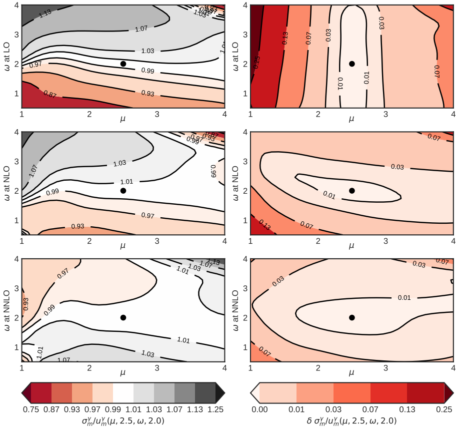

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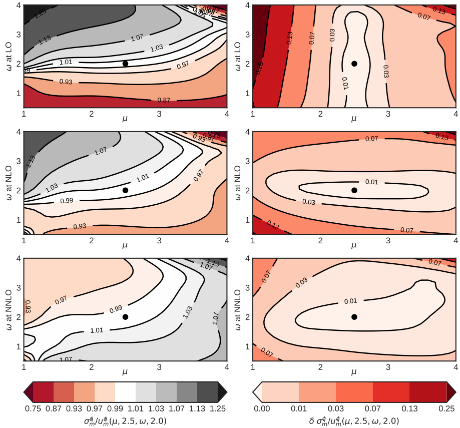
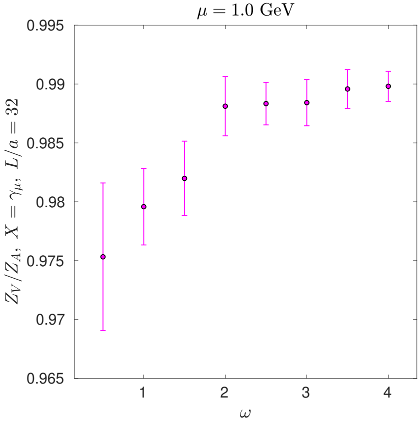


Study of the systematic effects

- Chiral symmetry breaking effects
- Vector Ward Identity
- Disctretisation effects

Results for Z_V/V_A



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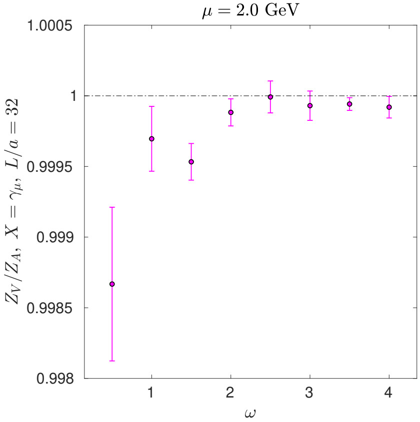
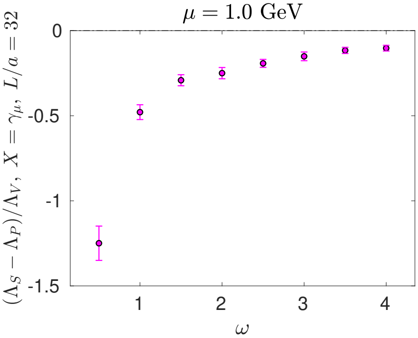


Results for $(\Lambda_S - \Lambda_P)/\Lambda_V$



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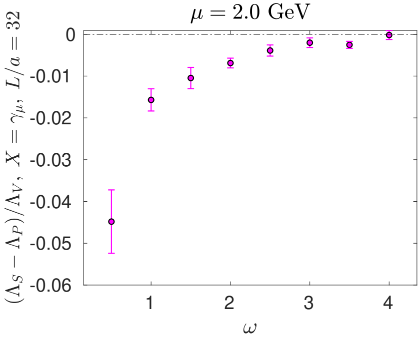
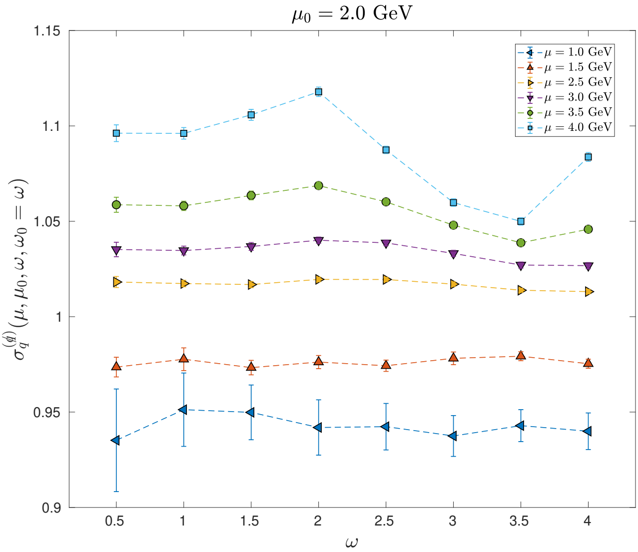
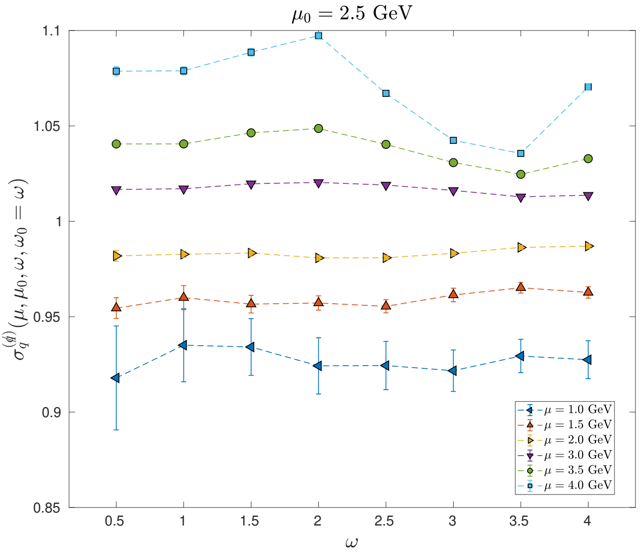


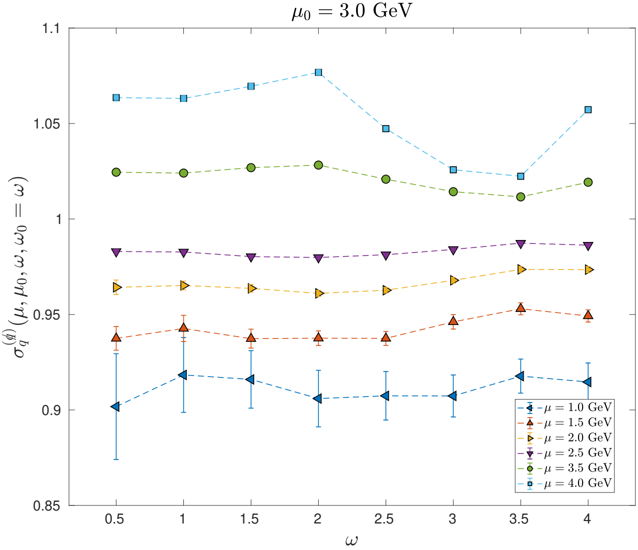
Results for $\sigma_q^{(\not{q})}$



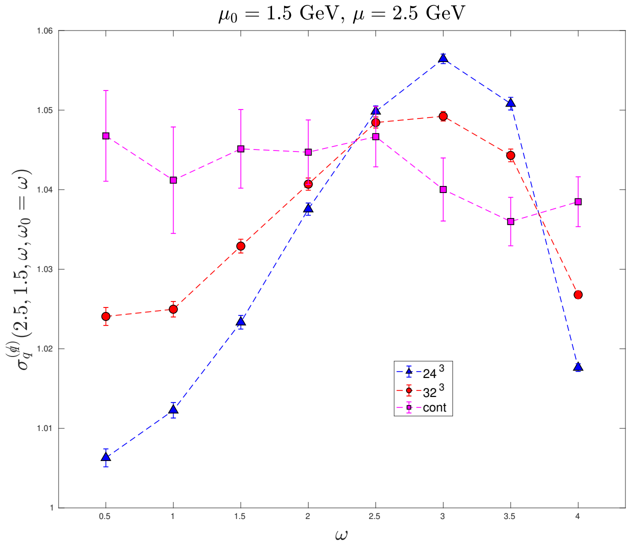
Results for $\sigma_q^{(\not{q})}$



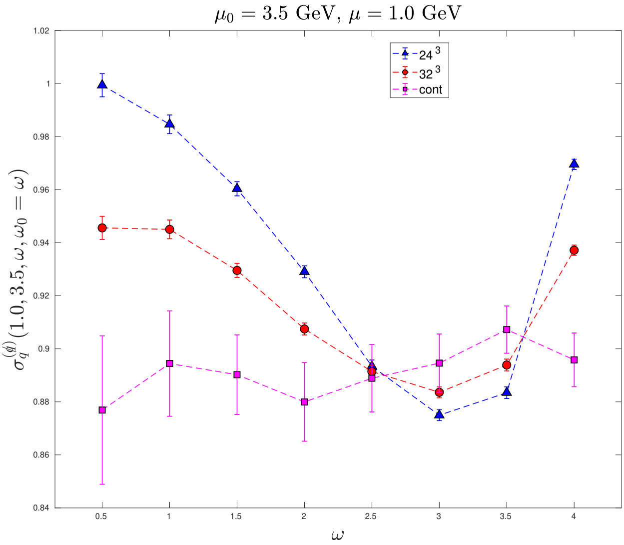
Results for $\sigma_q^{(\not{q})}$



Results for $\sigma_q^{(\not{q})}$ vs lattice spacing



Results for $\sigma_q^{(\not{q})}$ vs lattice spacing



Conclusions and outlook

- Proof of concept, first simulation of $\omega \neq 0, 1$
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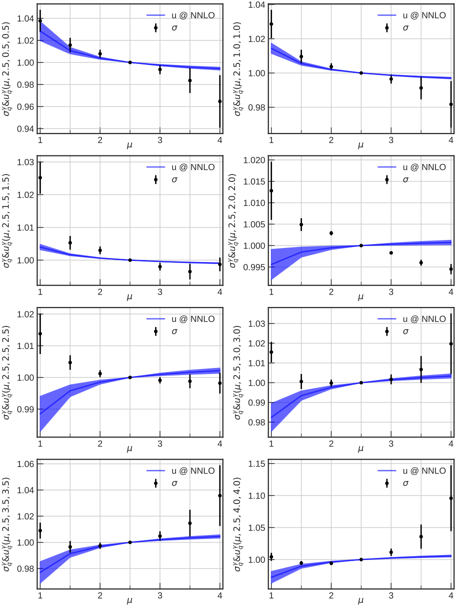
Proceedings and preprint (submitted to PRD)

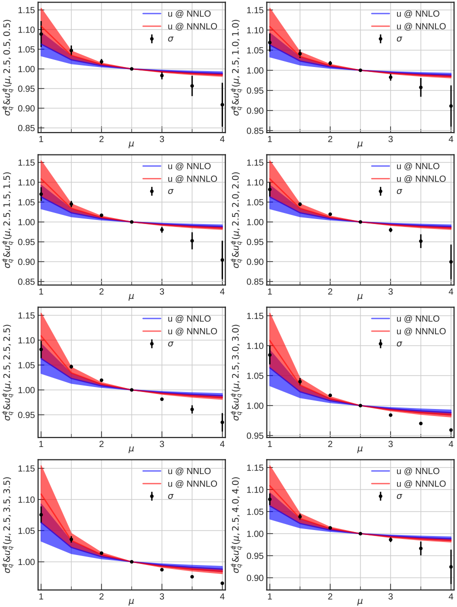
- `https://arxiv.org/abs/2202.04394`
- `https://arxiv.org/abs/2112.11140`

Backup

Results for $Z_q^{(\gamma_\mu)}$



Results for $Z_q^{(\not q)}$



What is going on for Z_q ?

Results for Z_q

Scheme	LO	NLO	NNLO	NNNLO	NP
$\overline{\mathrm{MS}}$	1.0	1.0048	1.0062	1.0064	
$\overline{\mathrm{MS}} \leftarrow \gamma_\mu$	1.0	1.0069	1.0078	N.A.	
$\overline{\mathrm{MS}} \leftarrow \not{q}$	1.0	1.0195	1.0175	1.0146	
γ_μ	1.0	1.0017	1.0020	N.A	1.0037(20)
$\not{q}$	1.0	1.0048	1.0081	1.0113	1.0195(25)

Table: Running between 2 and 2.5 GeV for the quark wave function in $\overline{\mathrm{MS}}$ and in the SMOM schemes $\gamma_\mu(\omega = 1)$ and $\not{q}$. In this case the running is known at NNNLO.

Results for Z_q

Scheme	NLO-LO	NNLO-NLO	NNNLO-NNLO
$\overline{\mathrm{MS}}$	0.0048	0.0013	0.0003
γ_μ	0.0017	0.0003	
$\not{q}$	0.0048	0.0033	0.0032

Table: Study of the convergence of the perturbative series for running of the quark wave function between 2 and 2.5 GeV in $\overline{\mathrm{MS}}$, SMOM-γ_μ and $\not{q}$.

Results for Z_m

Scheme	NLO-LO	NNLO-NLO	NNNLO-NNLO
$\overline{\mathrm{MS}}$	-0.0081	-0.0015	-0.0002
γ_μ	-0.0126	-0.0054	-0.0040
$\not{q}$	-0.0096	-0.0026	-0.0017

Table: Study of the convergence of the perturbative series for running of the quark mass between 2 and 2.5 GeV in $\overline{\mathrm{MS}}$, SMOM-γ_μ and $\not{q}$.

Results for Z_m

Scheme	LO	NLO	NNLO	NNNLO	NP
$\overline{\mathrm{MS}}$	0.9537	0.9456	0.9441	0.9439	
$\overline{\mathrm{MS}} \leftarrow \gamma_\mu$	0.9537	0.9350	0.9389	0.9426	
$\overline{\mathrm{MS}} \leftarrow \not{q}$	0.9537	0.9451	0.9462	0.9475	
γ_μ	0.9537	0.9411	0.9357	0.9318	0.9307(62)
$\not{q}$	0.9537	0.9441	0.9415	0.9400	0.9436(46)

Table: Running between 2 and 2.5 GeV for the quark mass.

Results for $Z_q^{(\gamma_\mu)}$

$\omega/\mu =$	1.0	1.5	2.5	3.0	3.5	4.0
0.5	0.972(8)	0.993(4)	1.008(4)	1.014(8)	1.023(14)	1.040(26)
1.0	0.976(8)	0.994(3)	1.004(2)	1.007(5)	1.012(8)	1.021(15)
1.5	0.978(4)	0.998(2)	1.003(1)	1.005(2)	1.006(3)	1.004(3)
2.0	0.990(7)	0.998(2)	1.003(0)	1.005(1)	1.007(1)	1.008(1)
2.5	0.987(5)	0.997(2)	1.001(1)	1.002(2)	1.002(3)	1.003(4)
3.0	0.985(4)	0.999(2)	1.000(2)	0.998(4)	0.993(9)	0.978(18)
3.5	0.989(5)	1.001(2)	0.997(2)	0.993(6)	0.982(13)	0.959(27)
4.0	0.990(5)	0.999(1)	0.994(3)	0.983(8)	0.957(22)	0.887(60)

Results for $Z_q^{(\not{q})}$

$\omega/\mu =$	1.0	1.5	2.5	3.0	3.5	4.0
0.5	0.935(27)	0.974(7)	1.018(7)	1.035(16)	1.059(31)	1.096(56)
1.0	0.951(19)	0.978(6)	1.017(6)	1.035(14)	1.058(28)	1.096(51)
1.5	0.950(15)	0.973(4)	1.017(4)	1.037(11)	1.064(25)	1.106(49)
2.0	0.942(15)	0.976(3)	1.020(3)	1.040(8)	1.069(20)	1.118(45)
2.5	0.942(14)	0.974(3)	1.019(1)	1.039(3)	1.060(9)	1.087(20)
3.0	0.937(12)	0.978(4)	1.017(2)	1.033(2)	1.048(2)	1.060(2)
3.5	0.943(10)	0.979(3)	1.014(2)	1.027(3)	1.039(3)	1.050(2)
4.0	0.940(10)	0.975(4)	1.013(3)	1.027(8)	1.046(18)	1.084(40)